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Non-uniqueness of the natural and projectively equivariant quantization

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Abstract

In [C. Duval, V. Ovsienko, Projectively equivariant quantization and symbol calculus: Noncommutative hypergeometric functions, Lett. Math. Phys. 57 (1) (2001) 61–67], the authors showed the existence and the uniqueness of a $sl(m+1,\mathbb{R})$-equivariant quantization in non-critical situations. The curved generalization of the $sl(m+1,\mathbb{R})$-equivariant quantization is the natural and projectively equivariant quantization. In [M. Bordemann, Sur l'existence d'une prescription d'ordre naturelle projectivement invariante (submitted for publication). math.DG/0208171] and [Pierre Mathonet, Fabian Radoux, Natural and projectively equivariant quantizations by means of Cartan connections, Lett. Math. Phys. 72 (3) (2005) 183–196], the existence of such a quantization was proved in two different ways. In this paper, we show that this quantization is not unique.

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1. Introduction

A quantization can be defined as a linear bijection from the space $\mathcal{S}(M)$ of symmetric contravariant tensor fields on a manifold M (also called the space of *Symbols*) to the space $\mathcal{D}_{\frac{1}{2}}(M)$ of differential operators acting between half-densities.

It is known that there is no natural quantization procedure. In other words, the spaces of symbols and of differential operators are not isomorphic as representations of $\mathrm{Diff}(M)$.

The idea of equivariant quantization, introduced by Lecomte and Ovsienko in [5], is to reduce the group of local diffeomorphisms in the following way.

They considered the case of the projective group $PGL(m+1,\mathbb{R})$ acting locally on the manifold $M=\mathbb{R}^m$ by linear fractional transformations. They showed that the spaces of symbols and of differential operators are canonically

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isomorphic as representations of $PGL(m+1,\mathbb{R})$ (or its Lie algebra $sl(m+1,\mathbb{R})$). In other words, they showed that there exists a unique *projectively equivariant quantization*. In [3], the authors generalized this result to the spaces $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$ of differential operators acting between λ- and μ-densities and to their associated graded spaces $\mathcal{S}_\delta$. They showed the existence and uniqueness of a projectively equivariant quantization, provided the shift value $\delta = \mu - \lambda$ does not belong to a set of critical values.

The problem of the $sl(m+1,\mathbb{R})$-equivariant quantization on $\mathbb{R}^m$ has a counterpart on an arbitrary manifold M. In [6], Lecomte conjectured the existence of a quantization procedure depending on a torsion-free connection, that would be natural (in all arguments) and that would remain invariant by a projective change of connection.

After the proof of the existence of such a *Natural and equivariant quantization* given by Bordemann in [1], we analysed in [7] the problem of this existence using Cartan connections. After these works, the question of the uniqueness of this quantization was not yet approached. The uniqueness of the $sl(m+1,\mathbb{R})$-equivariant quantization in the non-critical situations did not imply the uniqueness of the solution in the curved case. The aim of this paper is to show that this solution is not unique, even in non-critical situations, using the theory of Cartan connections.

2. Fundamental tools

For the sake of completeness, we briefly recall in this section the main notions and results of [7]. We refer the reader to this reference or to [4] for additional information. Throughout this note, we denote by M a smooth, Hausdorff and second countable manifold of dimension m.

2.1. Natural and projectively equivariant quantization

Denote by $\mathcal{F}_\lambda(M)$ the space of smooth sections of the vector bundle of λ-densities.

We denote by $\mathcal{D}_{\lambda,\mu}(M)$ the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$ and by $\mathcal{D}^k_{\lambda,\mu}$ the space of differential operators of order at most k. If $\delta = \mu - \lambda$, the associated space of *symbols* will be called $\mathcal{S}^k_\delta(M)$ and σ will represent the *principal symbol operator* from $\mathcal{D}^k_{\lambda,\mu}(M)$ to $\mathcal{S}^k_\delta(M)$.

In these conditions, a *quantization* on M is a linear bijection Q_M from the space of symbols $\mathcal{S}_\delta(M)$ to the space of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ such that

$$\sigma(Q_M(S)) = S, \quad \forall S \in \mathcal{S}^k_\delta(M), \forall k \in \mathbb{N}.$$

A *natural quantization* is a quantization which depends on a torsion-free connection and commutes with the action of diffeomorphisms.

More explicitly, if ϕ is a local diffeomorphism from M to N, then one has

$$Q_M(\phi^*\nabla)(\phi^* S) = \phi^*(Q_N(\nabla)(S)), \quad \forall \nabla \in \mathcal{C}_N, \forall S \in \mathcal{S}_\delta(N).$$

A quantization Q_M is *projectively equivariant* if one has $Q_M(\nabla) = Q_M(\nabla')$ whenever ∇ and ∇' are projectively equivalent torsion-free linear connections on M.

2.2. Projective structures and Cartan projective connections

We consider the group $G = PGL(m+1,\mathbb{R})$ acting on the projective space. We denote by H its isotropy subgroup at the origin. The group H is the semi-direct product $G_0 \rtimes G_1$, where G_0 is isomorphic to $GL(m,\mathbb{R})$ and G_1 is isomorphic to $\mathbb{R}^{m*}$. The Lie algebra associated with H is $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. The Lie algebra associated with G is then equal to $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1}$ is an abelian Lie subalgebra of $\mathfrak{g}$.

We recall that H can be seen as a subgroup of the group of 2-jets G^2_m.

A *projective structure on* M is then a reduction of the second-order frame bundle P^2M to the group H.

The following result [4, p. 147] is the starting point of our method:

Proposition 1 (*Kobayashi–Nagano*). *There is a natural one to one correspondence between the projective equivalence classes of torsion-free linear connections on M and the projective structures on M.*

We now refer the reader to [4] for the definition of a projective Cartan connection. Recall that if ω is a Cartan connection defined on an H-principal bundle P, then its curvature Ω is defined as usual by

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]. \tag{1}$$

We can define from Ω a function $\kappa \in C^\infty(P, \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g})$ by

$$\kappa(u)(X, Y) := \Omega(u)(\omega^{-1}(X), \omega^{-1}(Y)).$$

The *normal* Cartan connection has the following property (see [4, p. 136]):

$$\sum_i \kappa^i_{jil} = 0 \quad \forall j, \forall l.$$

Now, the following result [4, p. 135] gives the relationship between projective structures and Cartan connections:

Proposition 2. *A unique normal Cartan projective connection is associated with every projective structure P. This association is natural.*

The connection associated with a projective structure P is called the normal projective connection of the projective structure.

2.3. Lift of equivariant functions

If (V, ρ) is a representation of $GL(m, \mathbb{R})$, then we can define from it a representation (V, ρ') of H by projection (see [7] Section 3). If P is a projective structure on M, the natural projection $P^2M \to P^1M$ induces a projection $p : P \to P^1M$ and we have a well-known result:

Proposition 3. *If (V, ρ) is a representation of $GL(m, \mathbb{R})$, then the map*

$$p^* : C^\infty(P^1M, V) \to C^\infty(P, V) : f \mapsto f \circ p$$

defines a bijection from $C^\infty(P^1M, V)_{GL(m,\mathbb{R})}$ to $C^\infty(P, V)_H$.

Subsequently, we will use the representation ρ'_* of the Lie algebra of H on V. If we recall that this algebra is isomorphic to $gl(m, \mathbb{R}) \oplus \mathbb{R}^{m*}$ then we have

$$\rho'_*(A, \xi) = \rho_*(A), \quad \forall A \in gl(m, \mathbb{R}), \xi \in \mathbb{R}^{m*}. \tag{2}$$

Recall that $f \in C^\infty(P, V)_{G_1}$ if and only if

$$L_{h^*} f(u) = 0, \quad \forall h \in \mathbb{R}^{m*} \subset sl(m+1, \mathbb{R}), \forall u \in P. \tag{3}$$

Finally, we recall the definitions of two operators used subsequently:

Definition 1. Let (V, ρ) be a representation of H. If $f \in C^\infty(P, V)$, then $\nabla^{\omega^k}_s f \in C^\infty(P, S^k\mathbb{R}^{m*} \otimes V)$ is defined by

$$\nabla^{\omega^k}_s f(u)(X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\nu \in S_k} L_{\omega^{-1}(X_{\nu(k)})} \circ \cdots \circ L_{\omega^{-1}(X_{\nu(1)})} f(u).$$

If $(e_1, \ldots, e_m)$ is the canonical basis of $\mathbb{R}^m$ and if $(\epsilon^1, \ldots, \epsilon^m)$ is the corresponding dual basis in $\mathbb{R}^{m*}$, the *divergence operator* is defined by

$$\mathrm{Div}^\omega : C^\infty(P, S^k_\delta(\mathbb{R}^m)) \to C^\infty(P, S^{k-1}_\delta(\mathbb{R}^m)) : S \mapsto \sum_{j=1}^m \nabla^\omega_{e_j} S(\epsilon^j).$$

3. Non-uniqueness of the natural and projectively equivariant quantization

First, one makes the following remark:

Proposition 4. *A natural and projectively equivariant quantization Q_M is not unique if and only if there is a nonzero natural projectively equivariant application acting between $\mathcal{S}^k_\delta(M)$ and $\mathcal{S}^{k-l}_\delta(M)$ for one k and for one $l > 0$.*

Proof. A quantization being a bijection, the non-uniqueness of a natural projectively equivariant quantization is equivalent to the existence of two natural projectively equivariant quantizations Q and Q' and of a natural projectively equivariant application T from $\mathcal{S}_\delta(M)$ to $\mathcal{S}_\delta(M)$ different from the identity such that $Q' = Q \circ T$. There is at least one k such that the restriction of T to $\mathcal{S}^k_\delta(M)$ is different from the identity. As a quantization must preserve the principal symbol, the projection of this restriction on $\mathcal{S}^k_\delta(M)$ must be equal to the identity. The projections of the restriction on $\mathcal{S}^{k+l}_\delta(M)$, with $l > 0$, must be equal to zero and one can conclude. □

The construction of the applications discussed in the previous result is based on the Weyl tensor. Let us first recall its definition.

3.1. The Weyl tensor

If we denote by ω the normal Cartan connection associated with a projective structure P, the function κ induced by its curvature has an important property of invariance:

Proposition 5. *If $h \in H$, $u \in P$ and $X, Y \in \mathfrak{g}_{-1}$, the function $\kappa \in C^\infty(P, \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g})$ satisfies*

$$\kappa(X, Y)(uh) = Ad(h^{-1})\kappa(Ad(h)X, Ad(h)Y)(u). \tag{4}$$

The function κ_0, the component of κ with respect to $\mathfrak{g}_0$, is accordingly H-equivariant. It represents then a tensor of type $\binom{1}{3}$ on M that is called the Weyl tensor.

Proof. The relation (4) is proved in [2], page 44.

If one considers the components according to $\mathfrak{g}_0$ of the two sides of (4), one has

$$\kappa_0(X, Y)(uh) = \rho^{\mathbb{R}^m\otimes\mathbb{R}^{m*}}(h^{-1})\kappa_0(\rho^{\mathbb{R}^m}(h)X, \rho^{\mathbb{R}^m}(h)Y)(u),$$

where $\rho^{\mathbb{R}^m\otimes\mathbb{R}^{m*}}$ and $\rho^{\mathbb{R}^m}$ denote respectively the actions of H on $\mathbb{R}^m \otimes \mathbb{R}^{m*}$ and $\mathbb{R}^m$. Indeed, the components according to $\mathfrak{g}_{-1}$ of $Ad(h)X$ and $Ad(h)Y$ are respectively $\rho^{\mathbb{R}^m}(h)X$ and $\rho^{\mathbb{R}^m}(h)Y$. Moreover, the fact that $\kappa_{-1} = 0$ implies that the component according to $\mathfrak{g}_0$ of

$$Ad(h^{-1})\kappa(\rho^{\mathbb{R}^m}(h)X, \rho^{\mathbb{R}^m}(h)Y)(u)$$

is equal to

$$\rho^{\mathbb{R}^m\otimes\mathbb{R}^{m*}}(h^{-1})\kappa_0(\rho^{\mathbb{R}^m}(h)X, \rho^{\mathbb{R}^m}(h)Y)(u). \quad \square$$

3.2. Construction of natural and projectively equivariant applications

If $j \geq 2$ and if σ is a permutation of $\{1, \ldots, j\}$ such that $\sigma(l) \neq l\ \forall l \in \{1, \ldots, j\}$, we define an H-equivariant function $W \in C^\infty(P, S^{2j}\mathbb{R}^{m*})$ in the following way:

$$W(u)(e_{i_1}, \ldots, e_{i_{2j}}) := \sum_\nu \sum_{r_1,\ldots,r_j} \kappa_0(u)^{r_1}_{i_{\nu(1)}i_{\nu(2)}r_{\sigma(1)}} \ldots \kappa_0(u)^{r_j}_{i_{\nu(2j-1)}i_{\nu(2j)}r_{\sigma(j)}}.$$

The following lemma allows us to calculate the failure of equivariance of the iterated invariant differentials of the function W:

Lemma 6. *One has the following formula:*

$$L_{h^*}\nabla^{\omega^k}_s W = -k(k + 4j - 1)h \vee (\nabla^{\omega^{k-1}}_s W),$$

for all $h \in \mathfrak{g}_1$.

Proof. The proof is similar to the proof of the Proposition 10 of [7]. If $k = 0$, the formula is true. One proceeds then by induction. If $X \in \mathbb{R}^m$,

$$(L_{h^*}\nabla_s^{\omega^k} W)(X, \ldots, X)$$

is equal to

$$L_{\omega^{-1}(X)}L_{h^*}(\nabla_s^{\omega^{k-1}} W)(X, \ldots, X) + (L_{[h,X]^*}(\nabla_s^{\omega^{k-1}} W))(X, \ldots, X).$$

By induction, the first term is equal to

$$-(k-1)(k+4j-2)(h \vee (\nabla_s^{\omega^{k-1}} W))(X, \ldots, X).$$

As regards the second term, one obtains

$$(\rho_*((h \otimes X) + \langle h, X\rangle Id)(\nabla_s^{\omega^{k-1}} W))(X, \ldots, X).$$

The result comes then from the definition of ρ_*. □

One can then construct natural projectively equivariant applications between spaces of symbols:

Theorem 7. *If $S \in \mathcal{S}^k_\delta(M)$ and $l \geq 2j$, the multiples of the application*

$$S \mapsto p^{*^{-1}} \left(\sum_{r=0}^{l-2j} C_{k,l,r} \langle \mathrm{Div}^{\omega^r} p^* S, \nabla_s^{\omega^{l-r-2j}} W \rangle \right)$$

are natural and projectively equivariant if

$$C_{k,l,r} = \frac{(l+2j-1)!}{(m+1)^r (l+2j-1-r)! \gamma_{2k-1} \cdots \gamma_{2k-r}} \binom{l-2j}{r}, \quad \forall r \geq 1, C_{k,l,0} = 1.$$

Proof. The proof is similar to the proof of the Theorem 11 of [7]. Thanks to the Proposition 4 and to the Lemma 7 of [7], it suffices to check that the function

$$\sum_{r=0}^{l-2j} C_{k,l,r} \langle \mathrm{Div}^{\omega^r} p^* S, \nabla_s^{\omega^{l-r-2j}} W \rangle$$

is $\mathfrak{g}_1$-equivariant. This is true thanks to the previous lemma, to the Proposition 9 of [7] and to the fact that the following relation is satisfied:

$$C_{k,l,r} r(m + 2k - r - (m+1)\delta) = C_{k,l,r-1}(l - r - 2j + 1)(l - r + 2j).$$

The application given in the theorem is projectively equivariant by definition of ω. It is natural too: it follows from the naturality of the association of a projective structure $P \to M$ endowed with a normal Cartan connection ω with a class of projectively equivalent torsion-free connections on M. □

Remarks. • The applications that we have given are some examples of natural projectively equivariant applications between spaces of symbols. A complete description of the set of these applications seems however rather difficult.

- One can show "by hand" that the natural and projectively equivariant quantization is unique up to the third order in the non-critical situations. It suffices to consider all the natural applications between $\mathcal{S}^k(M)$ and $\mathcal{S}^{k-l}(M)$ (with $1 \leq l \leq 3$) and to show that there is no linear combination of these maps that is projectively equivariant in the non-critical situations.
- Using methods described in [2,8], one could derive explicit formulae for the applications of the Theorem 7. At the fourth and fifth orders, if we denote by T the equivariant function on P^1M corresponding to W with $j = 2$, the applications of the Theorem 7 are equal respectively to

$$\langle S, T\rangle$$

and to

$$\langle S, \nabla_s T\rangle + \frac{8}{(m+1)\gamma_{2k-1}} \langle \mathrm{Div} S, T\rangle.$$

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References

[1] M. Bordemann, Sur l'existence d'une prescription d'ordre naturelle projectivement invariante (submitted for publication). math.DG/0208171.
[2] A. Čap, J. Slovák, V. Souček, Invariant operators on manifolds with almost Hermitian symmetric structures. I. Invariant differentiation, Acta Math. Univ. Comenian. (NS) 66 (1) (1997) 33–69.
[3] C. Duval, V. Ovsienko, Projectively equivariant quantization and symbol calculus: Noncommutative hypergeometric functions, Lett. Math. Phys. 57 (1) (2001) 61–67.
[4] Shoshichi Kobayashi, Transformation Groups in Differential Geometry, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70.
[5] P.B.A. Lecomte, V.Yu. Ovsienko, Projectively equivariant symbol calculus, Lett. Math. Phys. 49 (3) (1999) 173–196.
[6] P.B.A. Lecomte, Towards projectively equivariant quantization, Progr. Theoret. Phys. Suppl. 144 (125–132) (2001) Noncommutative geometry and string theory (Yokohama, 2001).
[7] Pierre Mathonet, Fabian Radoux, Natural and projectively equivariant quantizations by means of Cartan connections, Lett. Math. Phys. 72 (3) (2005) 183–196.
[8] F. Radoux, Explicit formula for the natural and projectively equivariant quantization, Lett. Math. Phys. 78 (2) (2006) 173–188.